PUTNAM TRAINING PROBLEMS 2001.2
Oddball Mathematics

1. Show that there exist two irrational numbers a and b such that a^b is rational.

2. Prove that there exist infinitely many rational solutions (a, b) to the equation $a^b = b^a$ with $a \neq b$.

3. Prove that
$$\sum_{n=1}^{\infty} 6^{(2-3n-n^2)/2}$$
is irrational.

4. Find all positive integers which can be expressed as the sums of two or more consecutive positive integers.

5. Is
$$10^{5^{10^{5^{10}}}} + 5^{10^{5^{10^{5}}}}$$
divisible by 11?

6. A real polynomial $P(x)$ is bounded below. Prove that it necessarily attains its lower bound. What about polynomials $P(x, y)$ with two variables?

7. Prove or disprove: Two groups which are each isomorphic to a subgroup of the other must be isomorphic to each other.

8. Prove that $\mathbb{R}^3$ can be partitioned into loops. (A loop is a topological circle.)

9. Prove that if the top 26 cards of an ordinary shuffled deck contain more red cards than there are black cards in the bottom 26, then there are in the deck at least 3 consecutive cards of the same colour.