PUTNAM TRAINING PROBLEMS 2001.1
The Outer Limits

The Putnam competition has many problems involving limits. There is no general theory for tackling all these problems. You will just have to get used to working with continuity, approximations, ϵ's and such things. Keep in mind that

- for full marks it is not enough to naively evaluate a limit – your derivation must show that the limit exists as well.

1. For which values of the real number a does the series
$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\sin\frac{1}{n}\right)^{a}$$
converge?

2. Evaluate
$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right).$$

3. Evaluate
$$\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{1/n}.$$

4. Let $x_0=1$, and
$$x_{n+1}=\frac{3+2x_n}{3+x_n}.$$
Prove that $\lim x_n$ exists, and find its value.

5. Let $x_0=1$, and
$$x_{n+1}=\frac{1}{2+x_n}.$$
Prove that $\lim x_n$ exists and evaluate it.

6. Evaluate
$$\lim_{n\to\infty} n\int_0^1 x^n\, f(x)\,dx,$$
where f is continuous on the interval $[0,1]$.

7. Let k be a positive integer. Determine those real numbers c for which every sequence x_n, $n \geq 1$, of real numbers satisfying

$$\frac{x_{n+1} + x_{n-1}}{2} = c\,x_n$$

satisfies $x_{n+k} = x_n$ for all n.

8. Prove or disprove: Every infinite sequence of real numbers has either a nondecreasing subsequence or a nonincreasing subsequence.

9. **Putnam 1965 B1.** Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\left\{\frac{\pi}{2n}(x_1 \cdots + x_n)\right\} dx_1\, dx_2\, \cdots\, dx_n.$$

10. **Putnam 1965 B4.** Consider the function

$$f(x,n) = \frac{\binom{n}{0} + \binom{n}{2}x + \binom{n}{4}x^2 + \cdots}{\binom{n}{1} + \binom{n}{3}x + \binom{n}{5}x^2 + \cdots},$$

where n is a positive integer. Express $f(x, n+1)$ rationally in terms of $f(x,n)$ and x. Using this, or otherwise, evaluate $\lim_{n\to\infty} f(x,n)$ for suitable fixed values of x.

11. **Putnam 1981 A1.** Let $E(n)$ denote the largest integer k such that 5^k is an integral divisor of the product $1^1 2^2 3^3 \cdots n^n$. Calculate

$$\lim_{n\to\infty} \frac{E(n)}{n^2}.$$

12. **Putnam 1983 B5.** Let $||u||$ denote the distance from the real number u to the nearest integer. (For example, $||2.8|| = 0.2 = ||3.2||$.) For positive integers n, let

$$a_n = \frac{1}{n} \int_1^n \left|\left|\frac{n}{x}\right|\right| dx.$$

Determine $\lim_{n\to\infty} a_n$. You may assume the identity

$$\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9} \cdots = \frac{\pi}{2}.$$

13. **Putnam 1995 B4.** See handout.