PUTNAM TRAINING PROBLEMS 2000.1
Fun with Sums, Recursions & Stuff

1. Evaluate the following expressions:
   (a) $\sum_{n=0}^{\infty}(n+1)^2\,3^{-n}$
   (b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n-2}$
   (c) $\sum_{n=1}^{\infty}\frac{n^3}{n!}$

2. Prove that
$$\prod_{k=1}^{n-1}\sin\left(\frac{k\,\pi}{n}\right)=n\,2^{1-n}$$

3. (Putnam 1984) Express
$$\sum_{k=1}^{\infty}\frac{6^k}{(3^{k+1}-2^{k+1})\,(3^k-2^k)}$$
as a rational number.

4. Evaluate
$$\sum_{r=1}^{\infty}\left(\sum_{s=1}^{r}s^2\right)^{-1}$$

5. (Putnam 1976) Evaluate
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^n$$

6. (Putnam 1984) For each nonnegative integer k, let $d(k)$ denote the number of 1's in the binary expansion of k. (So $d(0)=0$ and $d(5)=2$.) Let m be a positive integer. Express
$$\sum_{k=0}^{2^m-1}(-1)^{d(k)}\,k^m$$
in the form $(-1)^m\,a^{f(m)}\,(g(m))!$ where a is an integer, and f and g are polynomials.