 Constant proportion portfolio insurance under a regime switching exponential Lévy process

Chengguo Weng * 
Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, N2L 3G1, Canada

HIGHLIGHTS
• The CPPI portfolio is studied with price jumps and market regime switching.
• Results are derived under general Markov’s regime switching Lévy models.
• Specific implementation is discussed under many popular Lévy models.
• Our results highlight the effects from the market state at the inception.

ABSTRACT
The constant proportion portfolio insurance is analyzed by assuming that the risky asset price follows a regime switching exponential Lévy process. Analytical forms of the shortfall probability, expected shortfall and expected gain are derived. The characteristic function of the gap risk is also obtained for further exploration on its distribution. The specific implementation is discussed under some popular Lévy models including the Merton’s jump–diffusion, Kou’s jump–diffusion, variance gamma and normal inverse Gaussian models. Finally, a numerical example is presented to demonstrate the implication of the established results.

1. Introduction
Portfolio insurance refers to those managing techniques designed to protect the value of a portfolio. They usually target to provide a guarantee on the terminal portfolio value by maintaining the portfolio value process not falling below a preset lower bound, which is called the floor. These techniques allow the investors to participate in equity market for its potential gains from an upside market move while limit the downside risk. The most prominent examples among the portfolio insurance strategies are the constant proportion portfolio insurance (CPPI) strategy and option-based portfolio insurance (OBPI) strategy. The OBPI combines a position in the risky asset with a put option on this asset; see for example El Karoui et al. (2005) and Leland and Rubinstein (1988) among many others.

The CPPI strategy involves no option. It adopts a simplified self-financing strategy to allocate capital between a risky asset (typically a traded fund or index) and a reserve asset (typically a bond) dynamically over time. In this method, the investor starts by setting a floor equal to the lowest acceptable value of the portfolio. Then, the investor computes the cushion as the excess of the portfolio value over the floor and allocates in the risky asset an amount of a constant multiple of the cushion. The constant is called multiplier. The amount allocated to the risky asset is known as the exposure, and the remainders are all invested in the reserve asset.
The CPPI strategy was initially introduced by Perold (1986) (see also Perold and Sharpe, 1988) for fixed-income instruments and Black and Jones (1987) for equity instruments. Extensive research has been conducted on CPPI in recent years, often either embedded in a more general framework or compared with other portfolio insurance strategies. A comparison of OBPI and CPPI (in continuous time) is given in Bertrand and Prigent (2005) and Balder and Mahanyi (2010); also see Do (2002) for an empirical investigation of both methods via simulation using Australian data. The performance of credit CPPI and constant proportion debt obligation structures is studied by Garcia et al. (2008) under a dynamic multivariate jump-driven model for credit spreads, and an investigation in much more depth under a similar setting can be found in Joossens and Schoutens (2010). The effect from price jumps on the performance of the CPPI strategy is studied by Cont and Tankov (2009) under a general exponential Lévy process. The literature also deals with stochastic volatility models and extreme value approaches on the CPPI method; see Bertrand and Prigent (2002, 2003). A general framework of CPPI for investment and protection strategies is formulated by Dersch (2010) along with proportional trading cost is investigated by Balder et al. (2010). The effectiveness of a CPPI portfolio and protection strategies is formulated by Dersch (2010) along with a review on other portfolio insurance techniques. The influence of estimation risk on the performance of CPPI strategies as well as the mitigation effect of the estimation risk by the robustification of mean–variance efficient portfolios is studied by Schöttle and Werner (2010). The effectiveness of a CPPI portfolio with proportional trading cost is investigated by Balder et al. (2009) under the Black–Scholes model and extended by Weng and Xie (2013) to a general exponential Lévy model. Moreover, a log-normal approximation approach for the gain of CPPI structure is also developed by Weng and Xie (2013).

When the trajectories of the price processes of both risky asset and reserve asset are continuous, the CPPI strategy with continuous trading will lead to a terminal portfolio value no less than the guaranteed value certainly, and hence fully achieve the purpose of portfolio insurance. Nevertheless, it has been widely noticed that there is always possibility for a CPPI portfolio to fall below the floor, leading to the notorious gap risk, which happens when the price of the risky asset drops substantially before the portfolio manager can rebalance the portfolio. Obviously, there are two main factors that may contribute to the gap risk: the illiquidity of the investment assets and the jump in the asset price. In this paper, we focus on the effect from the jump features of the asset price while presume that the investment assets are perfectly liquid.

The present paper is motivated by Cont and Tankov (2009), where, while the main results are derived under the exponential Lévy model assumption, the authors started with a semimartingale model setup and developed a general framework for evaluating the CPPI portfolio. The present paper aims to generalize the main results of Cont and Tankov (2009) to a regime switching exponential Lévy model. While the exponential Lévy process can well capture the jump feature in the price of financial assets, one of its obvious criticisms is its time homogeneity. In reality, the economic state usually shows an obvious feature of transition between two or among several states, and the financial return has quite different characteristics under a different economic state. As such, the regime switching model has been utilized widely nowadays; for its application in finance and actuarial science, see, for example, Buffington and Elliott (2002), Elliott et al. (1995, 2005), Hardy (2001), Li et al. (2008) and Siu (2005) among many others.

More specifically, in the present paper the dynamics of the asset prices are assumed to be governed by distinct exponential Lévy processes under different market states (e.g., bull and bear), and the transition from one market state to another is supposed to follow a hidden Markov process. The present paper obtains analytical forms for those important risk measures that are associated with the CPPI portfolio such as the short probability and the expected shortfall. The characteristic function of the shortfall is also obtained in an explicit form so as to make it possible to further explore on its distribution. In reality, the guarantee for the investors is usually provided by a bank (guarantor), which owns the CPPI portfolio, subject to a premium, and thus the gap risk is indeed assumed by the bank in exchange for the premium charged on the investors. Our established results will be helpful not only for the guarantor to conduct an effective evaluation on the gap risk and compute a reasonable level of premium but also for the investors to develop a good understanding on their risk-and-reward profile in investing a CPPI fund; for details, see the beginning part of Section 3.1. The specific implementation of our main results (and its challenges for some models) is investigated for some popular exponential Lévy processes including the Merton’s jump–diffusion, Kou’s jump–diffusion, variance gamma and normal inverse Gaussian models; see Section 4. Finally, our numerical example presented in Section 5 shows that the initial market state (bull or bear) of the investment will take a critical role in the resulting risk associated with a CPPI portfolio.

The rest of the paper is organized as follows. Section 2 is the model setup. The main results are collected in Section 3, following some preliminaries in the beginning of the section and followed by a discussion on how to derive the explicit formulas for the established main results. Section 4 investigates how to specifically implement main results for some popular exponential Lévy processes including the Merton’s jump–diffusion, Kou’s jump–diffusion, variance gamma and normal inverse Gaussian models. Section 5 presents a numerical example, where the effect of the regime switching feature of the market is demonstrated. Section 6 concludes the paper. Finally, proof of some lemmas and equations are relegated to the Appendix.

2. Model setup

Throughout the paper, we suppose that all the random elements involved are defined on a common probability space \((\Omega, F, \mathbb{P})\), and that the expectation of a random variable \(Z\) under \(\mathbb{P}\) is denoted by \(E(Z)\). The transpose of a matrix \(H\) (a vector \(a\)) will be denoted by \(H^\prime\) (\(a^\prime\)).

Assume that the state of the financial market is described by a finite state Markov process \(X := (X_t)_{t \geq 0}\). In particular, there could be just two states for \(X\), respectively, representing bull and bear; if we assume a third market state, it is typically interpreted as a normal state. As introduced by Elliott et al. (1995) and Buffington and Elliott (2002), we suppose that the Markov process \(X\) is generated by an intensity matrix \(Q\) with a finite state space of unit vectors \(e_1, \ldots, e_s\), where \(e_s = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n\) with 1 in its kth coordinate and 0 in all the others. According to Elliott et al. (1995), \(X\) has the following semimartingale representation

\[
X_t = X_0 + \int_0^t X_u Q du + M_t,
\]

where \(M\) is a martingale with respect to the filtration \(\mathcal{F}^X_t\) generated by \(X\).

Hereafter we assume that the CPPI portfolio is allocated between a stock index and a zero-coupon bond, and their price processes \(S\) and \(B\) are respectively subject to the following dynamics

\[
\frac{dS_t}{S_t} = dz_t \quad \text{and} \quad \frac{dB_t}{B_t} = dR_t,
\]

for two processes \(Z\) and \(R\) admitting the following regime switching structures

\[
Z_t = \sum_{j=1}^n (e_j) X_t (\omega_t^{(j)}), \quad \text{and} \quad R_t = \sum_{j=1}^n (e_j) R_t (\omega_t^{(j)}),
\]

with \(\omega_t \) being the hidden Markov process and \(\omega_t^{(j)}\) being the \(j\)-th component of \(\omega_t\).
where \((a, b)\) denotes the inner product \(\sum_{j=1}^{n} a_j b_j\) for two vectors \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\), and all \(Z_i^{(j)}\) and \(R_i^{(j)}\) are some semimartingales. The above specification implies that \(S\) and \(B\) are respectively driven by \(Z_t^{(j)}\) and \(R_t^{(j)}\) when the market is in the \(j\)th regime state \(e_j\) i.e., \(X_t = e_j\).

Let \(V_t\) denote the value of the CPPI portfolio at time \(t\) for \(t \geq 0\), and without any loss of generality, we assume that the floor is set as \((B_t)_{t \geq 0}\) so that the cushion at time \(t\) is computed as \(C_t = V_t - B_t\). At any time \(t\) with \(V_t > B_t\), the portfolio is maintained at an exposure of \(mC_t \equiv m(V_t - B_t)\) in stock, where \(m > 1\) is a constant multiplier; if \(V_t \leq B_t\), the entire portfolio is invested in the zero-coupon bond.

To analyze the CPPI portfolio, we borrow the same argument Cont and Tankov (2009) apply via the tools of the stochastic exponential and a change of numeraire. Let \(\tau := \inf\{t \geq 0 : V_t \leq B_t\}\) be the first time for which the CPPI portfolio value process \(V := (V_t)_{t \geq 0}\) touches or breaks through the preset floor \((B_t)_{t \geq 0}\). To avoid the trivial case with \(\tau = 0\), we assume \(V_0 > B_0\). Since the CPPI strategy is self-financing, up to time \(t\) the portfolio value satisfies

\[
dV_t = m(V_t - B_t) \frac{dS_t}{S_t} + (V_t - m(V_t - B_t)) \frac{dB_t}{B_t},
\]

which, by using the equality \(\frac{dC_t}{C_t} = m dZ_t + (1 - m) dR_t\), can be written as

\[
\frac{dC_t^*}{C_t^*} = \left(\frac{dZ_t}{Z_t} - d[R_t]_t - dR_t + d[R_t]_t\right),
\]

where \([\cdot, \cdot]\) and \([\cdot]\) respectively denote the quadratic covariation and the quadratic variation of the corresponding process(es). Let

\[
L_t := Z_t - [Z, R]_t - R_t + [R]_t.
\]

Then, we can rewrite Eq. (2.4) as

\[
C_t^* = C_0^* \delta (mL)_t,
\]

where \(\delta\) denotes the stochastic (Doléans–Dade) exponential defined by

\[
\delta (mL)_t = m dL_t.
\]

Therefore, the discounted cushion value at a general time \(t\) can be expressed as

\[
C_t^* = C_0^* \delta (mL)_{t \wedge \tau}, \quad t \geq 0,
\]

or alternatively

\[
\frac{V_t}{B_t} = 1 + \left(\frac{V_0}{B_0} - 1\right) \delta (mL)_{t \wedge \tau}, \quad t \geq 0.
\]

Hereafter we assume that \(Z_t^{(j)}\) and \(R_t^{(j)}\) in representation (2.3) are all Lévy processes, which are distinct at a different superscript. We refer the reader to Sato (1999), Applebaum (2004) and references therein for general theory on Lévy processes, and Schoutens (2003) and Cont and Tankov (2004) for their applications in finance modeling. From (2.5), \(L_t\) has the following regime switching structure

\[
L_t = \sum_{j=1}^{n} \left(\epsilon_j, X(t)\right) L_t^{(j)}
\]

with

\[
L_t^{(j)} = \left( Z_t^{(j)} - [Z^{(j)}, R^{(j)}]_t - R_t^{(j)} + [R^{(j)}]_t\right),
\]

which is obviously a Lévy process, \(j = 1, \ldots, n\). Assume that \(L_t^{(j)}\) has the following Lévy–Itô decomposition

\[
dL_t^{(j)} = b_t dt + \sigma_t dW_t + \int_{|x|>1} x(N_t(dx, dx))
\]

with a Lévy measure \(\nu_t\) and a corresponding Poisson random measure \(N_t\) so that \(N_t(A, t)\) is a Poisson process with intensity \(\nu_t(A)\) for any Borel set \(A \subset \mathbb{R} \setminus \{0\}\). The process \(L_t\) can be expressed in a more succinct way as follows:

\[
L_t = (L_t X_t), \quad \text{with } L_t = \left(I_t^{(1)}, \ldots, I_t^{(n)}\right).
\]

3. Gap risk and payoff of CPPI strategies

3.1. Preliminaries

Consider a CPPI portfolio invested over a finite time investment horizon \([0, T]\) with \(T > 0\), where the investors pay the initial value \(V_0\) at time 0 and are guaranteed to receive at least the value of the bond \(B_T\) at time \(T\). If the portfolio value \(V_T\) at time \(T\) is smaller than \(B_T\), a third party pays to the investor the shortfall amount \(B_T - V_T\). In practice, this guarantee is indeed usually provided by the bank which owns the CPPI portfolio, subject to a fee. Consequently, at the expiration date \(T\), the investors will receive a payoff of

\[
\max(V_T, B_T) \equiv B_T + \max(C_T, 0) \equiv B_T + C_T I_{\{C_T \leq 0\}},
\]

and the CPPI guarantor will be subject to a gap risk of \(C_T I_{\{C_T < 0\}}\), where \(I_{\{\cdot\}}\) is the indicator function. The evaluation on both quantities is interesting. An insightful investigation on the payoff can help the investors to establish a comprehensive understanding on their risk-and-reward profile in investing the CPPI portfolio, and a thorough analysis on the gap risk is necessary for the CPPI guarantor not only in computing a reasonable premium charged on the investors but also in their internal risk management. Obviously, both the gap risk of the guarantor and the payoff of the investors depend on the performance of bond, in addition to the stock. While it is interesting to conduct the investigation by taking into account the performance of both assets, in this paper we focus on the effect from the inherent jump features in the stock price and the regime switching mechanism of the financial market. Therefore, we measure both quantities at time 0 by discounting them using the bond price process \(B\), so that the time 0 value of the payoff for the investors is

\[
1 + \max\left\{\frac{C_T}{B_T}, 0\right\} \equiv 1 + \max\{C_T^*, 0\}
\]

and the time 0 value of the gap risk is \(\frac{C_T}{B_T} I_{\{C_T \leq 0\}}\). We further assume that the trajectories of the bond price process \(B\) are continuous so that all of these three events \(\{t \leq T\}, \{C_T \leq 0\}\) and \(\{C_T^* < 0\}\) are eventually the same, all indicating that a gap risk comes up during the investment time horizon \([0, T]\). As a result, the time 0 value of the gap risk is the same as

\[
\frac{C_T}{B_T} I_{\{C_T/B_T < 0\}} \equiv C_T^* I_{\{C_T^* \leq 0\}}.
\]

From Eqs. (3.9) and (3.10), we note that the time 0 values of both the payoff and the gap risk only depend on the discounted value of cushion \(C_T^*\). We shall focus on this quantity in our subsequent analysis.
To quantify the gap risk and payoff, we shall investigate the following quantities:

(i) the shortfall probability defined by
\[
SF(T) = \mathbb{P}(C_T^* < 0),
\]

(ii) the unconditional expected shortfall defined by
\[
UES(T) = \mathbb{E}(C_T^* 1_{C_T^* < 0}),
\]

(iii) the expected shortfall defined by
\[
ES(T) = \mathbb{E}(C_T^* | C_T^* \leq 0),
\]

(iv) the unconditional expected gain defined by
\[
UEG(T) = \mathbb{E}(C_T^* 1_{C_T^* > 0}).
\]

(v) the conditional expected gain defined by
\[
EG(T) = \mathbb{E}(C_T^* | C_T^* > 0).
\]

Besides, we will also work towards the distribution function of \(C_T^*\) by investigating its characteristic function.

**Remark 3.1.** It is obvious from Eqs. (2.4) and (2.5) that, when the driving process \(L\) is continuous, the loss probability \(SF(t)\) will be zero, and thus the gap risk must be resulted from the jump in the asset price if it comes up. This implies that the inherent Lévy measures \(\nu_j\) in process \(L\) will take a critical role in evaluating these measures defined above, as we can also see shortly.

**Remark 3.2.** If the investment horizon is not too long (say \(T = 1\) or 2) meaning one (or two) year), we may assume that the return rate associated with the bond is a constant \(r\). In this case, the expected shortfall and the expected gain of the time \(T\) quantity \(C_T\) can be computed by simply multiplying the corresponding measure with a factor of \(e^r\).

Hereafter, for \(t > 0\) and \(j = 1, \ldots, n\), we denote
\[
\Delta Y_j(t) := \int_0^t \{e_j, X_u\} du, \tag{3.11}
\]
the total amount of time which hidden Markov process \(X\) spends in state \(e_j\) over the time horizon \([0, t]\), i.e., the Lebesgue measure of the set \([u \in [0, t] : X_u = e_j]\). Clearly, we have \(\sum_{j=0}^{n} \Delta Y_j(t) = t\) almost surely, \(t > 0\). Borrowing the idea used in derivation of the characteristic function of \((G_1(t), \ldots, G_n(t))\) developed by Buffington and Elliott (2002), we have the following lemma.

**Lemma 3.1.** Let \(a := (a_1, \ldots, a_n)\) be a scalar vector, and \(X_t\) be the regime switching process defined in Eq. (2.1). Denote \(Y_t := \exp(\int_0^t a^T X_u du) X_t\). Then, for \(t \geq 0\),

(a) \(\mathbb{E}[Y_t] = \mathbb{E}[\exp(\sum_{j=0}^{n} a^T \Delta Y_j(t)) X_t] = \mathbb{E}(X_0)^\nu [\mathbb{E}(Q + \text{diag}(a)) t]\),

(b) \(\mathbb{E} \left( \exp \left( \int_0^t a^T X_u du \right) \right) = \mathbb{E}(X_0)^\nu (\mathbb{E}(Q + \text{diag}(a)) t))\),

(c) \(\mathbb{E} \left( \exp \left( \sum_{j=0}^{n} a^T \Delta Y_j(t) \right) \right) = \mathbb{E}(X_0)^\nu (\mathbb{E}(Q + \text{diag}(a)) t))\).

**Proof.** The proof is completely parallel with that of Lemma A.1 in Buffington and Elliott (2002) and hence omitted. \(\square\)

### 3.2 Shortfall probability

A CPPI portfolio incurs a shortfall (breaks through the floor), if \(V_t \leq B_t\) occurs for some \(t \in [0, T]\). Since the trajectories of the bond price process are assumed to be continuous, the event \(V_t \leq B_t\) is equivalent to \(C_t^* \leq 0\). Moreover, since \(\mathbb{E}(Y)_t = \mathbb{E}(Y)_{t-1} (1 + \Delta Y_t)\), \(C_t^* \leq 0\) for some \(t \in [0, T]\) if and only if \(m\Delta L_t \leq -1\) for some \(t \in [0, T]\). This leads to the following result.

**Proposition 3.1.** The shortfall probability \(SF(T)\) of the CPPI portfolio over an investment time horizon \([0, T]\) is given by
\[
SF(T) = 1 - A(T),
\]
where
\[
A(T) = \left\{ \begin{array}{ll}
\mu \in \mathbb{R}^n & \\
\mu_j = v_j((-\infty, -1/m]), & j = 1, 2, \ldots, n.
\end{array} \right.
\]

**Proof.** Let \(M\) be the number of transitions which the regime switching process \(X\) experiences over \([0, T]\). Denote \(\eta_0 = 0\) and \(\eta_{M+1} = t\), and let \(\eta_j\) be the \(j\)th transition time of the process for \(j = 1, 2, \ldots, M\). Let \(\tau_1, \tau_2, \ldots, \tau_{M+1}\) denote the corresponding inter-transition times so that \(\tau_j = \eta_j - \eta_{j-1}\) for \(j = 1, 2, \ldots, M + 1\). Let \(e_0\) denote the state in which the process \(X\) stays over period \((\eta_{j-1}, \eta_j), j = 1, 2, \ldots, M + 1\).

It is well known that the number of jumps of a Lévy process with Lévy measure \(\nu\) over a time horizon \((a, b)\) whose sizes fall in \((-\infty, -1/m)\) is a Poisson random variable with intensity \((b-a)\nu((-\infty, -1/m))\). Therefore, the conditional shortfall probability
\[
P \left( C_T^* \leq 0 | X_0, 0 \leq u \leq T \right)
\]
\[
= P(\exists t \in [0, T] : V_t \leq B_t | X_0, 0 \leq u \leq T)
\]
\[
= (1 - e^{-\tau_1 \mu_1}) + e^{-\tau_1 \mu_1} (1 - e^{-\tau_2 \mu_2})
\]
\[
+ \ldots + e^{-\tau_1 \mu_1 - \tau_2 \mu_2 - \ldots - \tau_M \mu_M} (1 - e^{-\tau_{M+1} \mu_{M+1}})
\]
\[
= 1 - \exp \left( - \sum_{j=1}^{M+1} \tau_j \mu_j \right).
\]

Consequently, the unconditional shortfall probability
\[
SF(T) = P(\exists t \in [0, T] : V_t \leq B_t)
\]
\[
= 1 - \mathbb{E} \left( \exp \left( - \sum_{j=1}^{M+1} \tau_j \mu_j \right) \right), \tag{3.13}
\]
Recall from (3.11) that \(\Delta Y_j(t)\) denotes the amount of time for which the process \(X\) spends in state \(e_j\) over \([0, T]\). Thus,
\[
\mathbb{E} \left( \exp \left( - \sum_{j=1}^{M+1} \tau_j \mu_j \right) \right) = \mathbb{E} \left( \exp \left( - \sum_{j=1}^{n} G_j(T) \mu_j \right) \right), \tag{3.14}
\]
which, along with (3.13) and part (c) of Lemma 3.1, yields the desired result. \(\square\)

**Corollary 3.1.** The density of \(\tau \equiv \inf\{t \geq 0 : V_t \leq B_t\}\), denoting the moment for the CPPI portfolio value hits or falls below the floor, is given by
\[
\frac{d}{dt} f_t(t) = - (1, (\mathbb{E}(X_0)) \exp((Q - \text{diag}(\mu)) t)) \mathbb{E} \left( (Q - \text{diag}(\mu)) \right) t) \mu, \quad \text{for } t \geq 0.
\]

**Proof.** The proof follows directly from Proposition 3.1 and the fact that \(SF(t) = P(\tau \leq t)\). \(\square\)

**Remark 3.3.** It is interesting to note that the results in Proposition 3.1 and Corollary 3.1 are heuristically obvious, if we recall the definition of the so-called phase-type distribution.

Let \((Y_t)_{t \geq 0}\) be a Markov jump process with a finite state space \(E = \{e_1, \ldots, e_{n+1}\}\), where states \(e_1, \ldots, e_n\) are transient and state \(e_{n+1}\) is absorbing. Then, \((Y_t)_{t \geq 0}\) has an intensity matrix of the form
\[
\begin{pmatrix}
P_{00} & 0 \\
0 & P_{00}
\end{pmatrix}.
\]
where \( \mathbf{P} \) is a matrix of \( n \) by \( n \), \( \mathbf{p} \) is a \( n \) dimensional column vector and \( \mathbf{0} \) is the \( n \) dimensional row vector of zeros. Note that we must have \( \mathbf{p} = -\mathbf{P} \mathbf{1} \) since each row of an intensity matrix must sum to 0. Let \( \xi \) be the time until absorption, i.e., \( \xi = \inf \{ t \geq 0, Y_t = e_{n+1} \} \). The distribution of \( \tau \) is said to have a phase-type distribution, written as \( \tau \sim PH(\pi, \mathbf{P}) \), where \( \pi = (\pi_1, \ldots, \pi_n) \) with \( \pi_j = \mathbb{P}(Y_0 = e_j) \) for \( j = 1, \ldots, n \). It is well known that the phase-type distribution has a presentation of

\[
\mathbb{P}(\tau \leq t) = (1 \times \pi \mathbb{P} \exp(\mathbf{P}t)),
\]

with a density function

\[
f_\tau(t) = (1 \times \pi \mathbb{P} \exp(\mathbf{P}t)).
\]

The phase-type distributions have been widely used as one of the important modeling tools in finance and risk theory; see, for example, Bladt (2005) and Asmussen (2000).

In the context of the CPPI portfolio which we analyzed in Proposition 3.1, we may let \( Y_t \) be the same as the regime switching process \( X_t \) all the time before the portfolio value breaks the floor, and set it equal to \( e_{n+1} \) thereafter. Then, the event \( \{ Y_t = e_{n+1} \} \) means that a shortfall happens before or at time \( t \), and thus \( \tau = \inf \{ Y_t = e_{n+1} \} \). In other words, the portfolio incurs a shortfall if and only if \( Y_t \) is absorbed by the state \( e_{n+1} \). Recall from the paragraph right before Proposition 3.1 that the CPPI portfolio incurs a shortfall if and only if a downward jump in the process \( X_t \) happens with a size larger than \( 1/m \). Conditional on the regime \( e_i \), we have a loss of such a size is \( \mu_j \equiv \psi_j(-\infty, -1/m) \) for \( j = 1, \ldots, n \). This implies that \( Y_t \) has an intensity matrix

\[
\begin{pmatrix}
-Q & \text{diag}(\mu) \\
0 & 0
\end{pmatrix},
\]

and consequently by applying the presentations in (3.15) and (3.16) we immediately recover the results in Proposition 3.1 and Corollary 3.1.

3.3 Expected shortfall and expected gain

For each Lévy process \( L^j \), we can always write \( L^j = L^{(1)} + L^{(2)} \), where \( L^{(2)} \) is a process with piecewise trajectories and jumps satisfying \( \Delta L_{\tau}^{(2)} \leq -1/m \) and \( L^{(1)} \) is a process with jumps satisfying \( \Delta L_{\tau}^{(1)} \geq -1/m \). In other words, \( L^{(2)} \) has Lévy measure \( \psi_{\tau}(dx) \) with no diffusion component, and no drift. The jump intensity of \( L^{(2)} \) is

\[
\mu_j = \psi_j(-\infty, -1/m), \quad j = 1, \ldots, n.
\]

Let \( \phi_j(\tau) \) be the characteristic function of the Lévy process \( \varepsilon (ml_{\tau}^{(1)}), \psi_j(u, t) : \tau \rightarrow t \) be the corresponding Lévy exponent. For presentation convenience, we will use the following notation

\[
\theta_j := \phi_j(1) = \frac{1}{t} \ln \mathbb{E} \left( \varepsilon (ml_{\tau}^{(1)}), \right), \quad j = 1, \ldots, n.
\]

Further denote \( L_{\tau}^{(1)} = (l_{\tau}^{(1)}, \ldots, l_{\tau}^{(n)}) \) and \( L_{\tau}^{(2)} = (l_{\tau}^{(1)}, \ldots, l_{\tau}^{(n)}) \), let \( L_{\tau}^{(1)} \) and \( L_{\tau}^{(2)} \) respectively denote \( (l_{\tau}^{(1)}, \mu) \) and \( (l_{\tau}^{(2)}, \mu) \) so that \( L_{\tau} = l_{\tau}^{(1)} + l_{\tau}^{(2)} \), and use \( \Delta L_{\tau}^{(2)} \) to denote \( l_{\tau}^{(2)} - l_{\tau}^{(2)} \), the jump of \( L_{\tau}^{(2)} \) at time \( t \). Moreover, we use \( \kappa_j \) to denote \( \mathbb{E} \left( 1 + ml_{\tau}^{(2)} \right) \) so that

\[
\kappa_j = 1 + \frac{m}{\mu_j} \int_{x < -1/m} x \psi_j(dx), \quad j = 1, \ldots, n.
\]

Finally, let \( \kappa : (\kappa_1, \ldots, \kappa_n) \).

To proceed, we recall some useful properties of the Lévy process as given below.

Lemma 3.2. (a) Let \( (U)_{\tau \geq 0} \) be a real valued Lévy process with Lévy triplet \((\sigma^2, \nu, \omega)\) and \( Y = \varepsilon(U) \) its stochastic exponential.

If \( Y > 0 \) a.s., then there exists another Lévy process \( (\tilde{U})_{\tau \geq 0} \) satisfying \( Y_t = e^{\tilde{U}_t} \) and

\[
\tilde{U}_t = U_t - \frac{\sigma^2 t}{2} + \sum_{0 \leq s \leq t} (\ln(1 + DU_s) - DU_s).
\]

The Lévy triplet \((\tilde{\sigma}^2, \tilde{\nu}, \tilde{\omega})\) of \( \tilde{U} \) is given by

\[
\tilde{\sigma} = \sigma, \quad \tilde{\nu} = \omega - \frac{\sigma^2}{2} + \int (\ln(x+1)) [1 - \ln(1+x)] dv(x), \quad \tilde{\omega} = \omega + \frac{\sigma^2}{2} + \int (\ln(x+1)) [1 + \ln(1+x)] v(dx).
\]

(b) Let \( (\tilde{U})_{\tau \geq 0} \) be a real valued Lévy process with Lévy triplet \((\tilde{\sigma}^2, \tilde{\nu}, \tilde{\omega})\) and \( Y_t = e^{\tilde{U}_t} \) its exponential. Then there exists another Lévy process \( (U)_{\tau \geq 0} \) with \( Y_t = \varepsilon(U) \) and

\[
U_t = \tilde{U}_t + \frac{\tilde{\sigma}^2 t}{2} + \sum_{0 \leq s \leq t} \left( 1 + \Delta \tilde{U}_s - e^{\Delta \tilde{U}_s} \right).
\]

The Lévy triplet \((\sigma^2, \nu, \omega)\) of \( U \) is given by \( \sigma = \tilde{\sigma}, \nu = \tilde{\nu} - \int (e^x - 1) \tilde{v}(dx), \omega = \tilde{\omega} + \tilde{\sigma}^2/2 + \int ((e^x - 1)^2 - (e^x - 1)(1 - \ln(1+x))) \tilde{v}(dx).\)

(c) \( \mathbb{E} \left[ \varepsilon (ml_{\tau}^{(2)}) \right] = \phi_j(-\tau; t) = e^{\theta_j t} \), where

\[
\theta_j = \mu_j t + \int_{x < -1/m} x \psi_j(dx), \quad j = 1, \ldots, n.
\]

where the constant \( \mu_j \) and Lévy measure \( \psi_j \) are from Eq. (2.8).

Proof. For the proof of (a) and (b), see Cont and Tankov (2004, Chapter 9) or Gold and Kallsen (2000); also see part 1 of Proposition A.1 from Cont and Tankov (2009). (c) follows from part (a) and Theorem 25.17 in Sato (1999, p. 165).

Proposition 3.2. Denote \( \theta := (\theta_1, \ldots, \theta_n) \), where \( \theta_i \) is given in (3.20). Then, we have the following results regarding the expected shortfall and expected gain of the CPPI portfolio.

(a) The unconditional expected shortfall is

\[
\mathbb{E} \left[ C^*_T | C^*_T \leq 0 \right] = \int_0^T B(t) C(t) dt, \quad \text{and hence, the expected shortfall is}
\]

\[
\mathbb{E} \left[ C^*_T | C^*_T \leq 0 \right] = \int_0^T B(t) C(t) dt / (1 - A(T)).
\]

(b) The unconditional expected gain

\[
\mathbb{E} \left[ C^*_T | C^*_T > 0 \right] = \left( \mathbb{E}(X_0)^+ \right) \exp \{ (Q + \text{diag}(\theta)) t \} A(T), \quad \text{and the conditional expected gain}
\]

\[
\mathbb{E} \left[ C^*_T | C^*_T > 0 \right] = \left( \mathbb{E}(X_0)^+ \right) \exp \{ (Q + \text{diag}(\theta)) t \} A(T).
\]
Proof. (a) For any time \( t \in [0, T] \), the discounted cushion satisfies

\[
C_t^* = \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \\
= \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} + \mathbb{E} \left( \overline{m}^{(2)} \right)_{t, \tau} \left( 1 + m \Delta_t \right).
\]

(3.25)

As in proof of Proposition 3.1, we let \( M \) be the number of transitions which the regime switching process \( X \) experiences over \([0, t]\), let \( \eta_0 = 0 \) and \( \eta_{M+1} = t \), and let \( \eta_j \) denote the \( j \)th transition time of the regime switching process for \( j = 1, \ldots, M \). Let \( E_j \) denote the state in which the process \( X \) stays over period \([\eta_{j-1}, \eta_j)\), \( j = 1, 2, \ldots, M + 1 \). Let \( \tau_1, \tau_2, \ldots, \tau_{M+1} \) denote the corresponding inter-transition times so that \( \tau_j = \eta_j - \eta_{j-1} \) for \( j = 1, 2, \ldots, M + 1 \).

Then, we have the following expression for the stochastic exponential:

\[
\mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} = \prod_{j=1}^{M+1} \mathbb{E} \left( \overline{m}^{(1)} \right)_{\eta_j, \eta_{j-1}}
\]

(3.26)

where \( \overline{m}^{(1)} \) is identical to the Lévy process \( m^{(1)} \) for \( t \in [\eta_{j-1}, \eta_j)\), \( j = 1, \ldots, M + 1 \). According to Lemma 3.2, for each \( j \), there exists a Lévy process \( \overline{m}^{(2)} \) such that \( \mathbb{E} \left( \overline{m}^{(2)} \right) = e^{\nu \tau} \) so that all the items in the product (3.26) are independent conditional on \( \mathcal{F}_t \) and

\[
\mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{\eta_j, \eta_{j-1}} \bigg| \mathcal{F}_t \right) = \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{\eta_j, \eta_{j-1}} \bigg| \eta_{j-1}, \eta_j, E_j \right) = \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{\eta_j, \eta_{j-1}} \bigg| \eta_{j-1}, \eta_j \right) = \exp \left( \eta_j \frac{\nu_j(K_j)}{\nu_{j-1}(K_{j-1})} - V_{\eta_j(K_j)} - V_{\eta_{j-1}(K_{j-1})} \right)
\]

(3.27)

where \( K_j \) is a random variable such that \( E_j = e_{K_j} \). Moreover, by the property of the Lévy process, \( \mathbb{E} \left( \overline{m}^{(1)} \right) = e^{\nu \tau} \) is independent of \( \Delta_t^{(2)} \) conditional on \( \mathcal{F}_t \), which is the \( \sigma \)-algebra generated by \( \{X_u, 0 \leq u \leq t\} \), and indeed the latter quantity only depends on \( X_t \).

The above analysis implies that

\[
\mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right) = \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right) \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right)
\]

(3.28)

Note that, conditional on \( \mathcal{F}_t \), the increment \( V_{\eta_j(K_j)} - V_{\eta_{j-1}(K_{j-1})} \) has the same distribution as \( V_{\eta_j} - V_{\eta_{j-1}} \), and all of these increments are independent of each other. Thus, from (3.28) it follows that

\[
\mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \right) = \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right) \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right)
\]

(3.29)

where the last step is due to part (b) of Lemma 3.2. We further note the following inequality

\[
\sum_{j=1}^{M+1} \tau_j \theta_j = \sum_{j=1}^{n} \Gamma_j(t) \theta_j.
\]

Thus, by applying part (a) of Lemma 3.1, we derive from (3.29) that

\[
\mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \right) = \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right) \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right) = \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \right).
\]

(3.30)

and consequently, it follows from the density of \( \tau \) given in Corollary 3.1 and representation of \( C_t^* \) in (3.25) that

\[
\mathbb{E} [C_t^* | C_{t'} > 0] = \mathbb{E} [C_t^* | C_{t'} > 0] \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right) = \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right) = \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right) = \mathbb{E} \left( \mathbb{E} \left( \overline{m}^{(1)} \right)_{t, \tau} \left( 1 + m \Delta_t \right) \bigg| \mathcal{F}_t \right).
\]

(3.31)

Finally, by noticing that \( \mathbb{P}(\tau > T) = A(T) \), we obtain the desired results. \( \square \)

3.4. Loss distribution

Let \( \tilde{\phi}_j(u) := \frac{1}{\nu_j} \int_{-\infty}^{-1/m} e^{u \mu_j n(-\infty, -1/m)} dx \) denoting the characteristic function of \( \ln (1 - m \Delta_t^{(2)}) \), where we recall that \( \mu_j \) denotes \( \nu_j((-\infty, -1/m]) \), the jump intensity of \( L^{(2)}_j \), \( j = 1, \ldots, n \). Denote \( \phi_j(u) := \tilde{\phi}_j(u) \). Also recall that \( \psi_j(u) \) denotes the characteristic exponent of \( \ln \left( m^{(1)} \right) \), so that

\[
\mathbb{E} \left( e^{u \mu_j n \left( \ln \left( m^{(1)} \right) \right)} \right) = e^{\psi_j(u) \mu_j n}, \quad u \in \mathbb{R}.
\]

\[
\Psi(u) := (\psi_1(u), \ldots, \psi_n(u))', \quad u \in \mathbb{R}.
\]
The next proposition provides a possible way to explore the distribution of the gap risk by using the Fourier inversion technique.

**Proposition 3.3.** (a) The characteristic function of the shortfall \((-C^*_T)\) conditional on the event \((C^*_T < 0)\) is given by

\[
\psi(u) := E\left(e^{iu \ln(-C^*_T)} | C^*_T < 0 \right) = \frac{\int_0^T C(t)D(t, u)dt}{1 - A(T)},
\]

where

\[
D(t, u) = \left\{ \phi(u), \ (\mathbb{E}(X_0))' \exp\{(Q + \operatorname{diag} (\psi(u)))t\} \right\}. \tag{3.31}
\]

(b) Choose a random variable \(X^*\) with a characteristic function \(\phi^*\) such that \(E|X^*| < \infty\) and \(\frac{d}{d\psi(u)}\) is integrable. If, additionally, \(\int_0^\frac{1}{1 + \lvert u \rvert} \int_0^T C(t)D(t, u)dt \]

is integrable, and \(\int_{|x| \geq \epsilon} (\lvert \ln(1 + nx) \rvert) v_j(dx) < \infty\) for sufficiently small \(\epsilon\), \(j = 1, \ldots, n\), then for every \(x < 0\),

\[
\mathbb{P}(C^*_T \leq x | C^*_T < 0) = \frac{\frac{1}{2\pi} \mathbb{E}\left( e^{-iu \ln(-x)} \times \left( \int_0^T \frac{C(t)D(t, u)dt}{iu(1 - A(T))} - \frac{\phi^*(u)}{iu} \right) du. \right)}{\mathbb{E}\left( e^{iu \ln(-C^*_T)} | C^*_T < 0 \right)}.
\]

**Proof.** (a) By (3.25) and Proposition 3.1, we have

\[
E\left( e^{iu \ln(-C^*_T)} | r \leq T \right) = E\left( e^{iu \ln(-C^*_T)} | C^*_T \leq 0 \right) = \frac{E(T)}{1 - A(T)}, \tag{3.32}
\]

where

\[
E(T) = \int_0^T E\left( e^{iu \ln(-t \ln(1 + ntx))} f_j(t) \right) dt. \tag{3.33}
\]

Using the conditional independence between \(\epsilon(t \ln(1 + ntx))\) and \(\tilde{L}^2(t)\), we can calculate the expectation in the above integral as follows.

\[
E\left( e^{iu \ln(-t \ln(1 + ntx))} \right) = \mathbb{E}\left( \mathbb{E}\left( e^{iu \ln(-t \ln(1 + ntx))} | f_j(t) \right) \right).
\]

Using the same technique we applied in the proof of Proposition 3.2, we obtain

\[
E\left( e^{iu \ln(-t \ln(1 + ntx))} \right) = \prod_{j=1}^n \mathbb{E}\left( e^{iu \ln(-t \ln(1 + ntx))} \right) = \prod_{j=1}^n \mathbb{E}\left( e^{iu \ln(-t \ln(1 + ntx))} \right) \cdot \mathbb{E}\left( e^{iu \ln(-t \ln(1 + ntx))} \right).
\]

Thus, it follows from part (a) of Lemma 3.1 that

\[
E\left( e^{iu \ln(-t \ln(1 + ntx))} \right) = E\left( \phi(u), \ (\mathbb{E}(X_0))' \exp\{(Q + \operatorname{diag} (\psi(u)))t\} \right).
\]

(b) The proof follows directly from applying Lemma B.1 of Cont and Tankov (2009). Also see the proof of Theorem 3.1 in Cont and Tankov (2009). \(\square\)

### 3.5. Analysis under a diagonalizable intensity matrix

The results established in the previous subsections depend on these functions \(A(t), B(t), C(t)\) and \(D(t, u)\) defined in Propos- itions 3.1–3.3, where matrix exponentials are involved; see Eqs. (3.12), (3.22), (3.23) and (3.31). To evaluate the CPPI portfolio by using these results, we have to make sure that these functions are computationally amenable. While there are many ways to calculate or approximate the matrix exponentials, see Moler and Loan (2003) for example, we adopt Putzer’s spectral formula (see Putzer, 1996). Indeed, because the intensity matrix \(Q\) associated with a regime switching process is always diagonalizable with distinct eigenvalues to be meaningful, Putzer’s spectral formula turns out to be very friendly in developing explicit forms of the functions \(A(t), B(t), C(t), D(t, u)\) and those related measures we target.

As we can see shortly, Putzer’s spectral formula is an accurate approach in calculating a matrix exponential, though the computational efforts increase exponentially along with the dimension of the exponent matrix. Therefore, it is an ideal way only in dealing with the low-dimensional case. Fortunately, in the financial context, a very small number of regime states are often sufficient to capture the state transition of a financial market. Indeed, the two-state regime switching model is the most common one with states respectively representing a “bear” market and a “bull” market. As mentioned in the very beginning of Section 2, when three states are assumed in the model, the third one means a “normal” state of the market. Hereafter we assume that the intensity matrix \(Q\) has distinct eigenvalues \(r_1, \ldots, r_n\).

**Lemma 3.3.** Let \(\lambda_1, \ldots, \lambda_n\) be the eigenvalues of a matrix \(H\) of \(n\) by \(n\), which are not necessarily distinct, in some arbitrary but specified order. Let \(I\) denote the identity matrix of an appropriate dimension. Then,

\[
e^{Hu} = \sum_{j=0}^{n-1} f_{j+1}(t)E_j,
\]

where \(P_0 = I, P_j = \prod_{i=1}^{j} (H - \lambda_i I), \ j = 1, \ldots, n, \) and \((r_1, \ldots, r_n(t))\) is the solution to the following triangular system

\[
\begin{align*}
\frac{d}{dt} r_j(t) &= \lambda_j r_j(t), \\
\frac{d}{dt} r_j(t) &= r_{j-1}(t) + \lambda_j r_j(t), \quad j = 2, \ldots, n, \\
r_1(0) &= 1, \quad r_j(0) = 0, \quad j = 2, \ldots, n.
\end{align*}
\]

**Proof.** See Putzer (1996). \(\square\)
Obviously, system (3.36) can be solved explicitly provided that all the eigenvalues \( \lambda_i \) are distinct, although [Putzer, 1996] refrains from doing so. We summarize its explicit solutions in the following lemmas.

**Lemma 3.4.** When all the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( H \) are distinct, the solutions to the system (3.36) are as follows:

\[
\begin{align*}
\mathbf{r}_n(t) &= \sum_{j=1}^{n} \left( e^{\lambda_j t} \prod_{k=1, k \neq j}^{n} \frac{1}{\lambda_j - \lambda_k} \right), \quad n \geq 1. \\
\text{Proof.} &\quad \text{See Appendix.} \quad \square
\end{align*}
\]

By applying Lemmas 3.3 and 3.4, we can obtain the explicit form for the matrix exponential in the case of distinct eigenvalues, as shown in the following lemma.

**Lemma 3.5.** Assume that a matrix \( H \) of \( n \times n \) has distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \). Then, the matrix exponential \( e^{Ht} \) has the following representation:

\[
e^{Ht} = \sum_{j=1}^{n} \left( e^{\lambda_j t} \prod_{k=1, k \neq j}^{n} \frac{H - \lambda_j I}{\lambda_j - \lambda_k} \right).
\] (3.38)

**Proof.** See Appendix. \( \square \)

Now we can readily derive the explicit forms for functions \( A(t), B(t), C(t) \) and \( D(t, u) \), which are respectively defined in (3.12), (3.22), (3.23) and (3.31).

**Proposition 3.4.** Suppose that the intensity matrix \( \mathbf{Q} \) associated with the regime process \( \mathbf{X}_t \) in (2.1) has distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \). For \( j = 1, \ldots, n \), denote

\[
(\beta_{j,1}, \ldots, \beta_{j,n}) := (\mathbb{E}(X_0))^j \prod_{k=1, k \neq j}^{n} \frac{Q_{ij} - \lambda_j I}{\lambda_j - \lambda_k}, \quad j = 1, \ldots, n.
\]

Then,

\[
\begin{align*}
A(t) &= \sum_{j=1}^{n} \sum_{i=1}^{n} \beta_{j,i} e^{(\lambda_j - \mu_j)t}, \\
B(t) &= \sum_{j=1}^{n} \sum_{i=1}^{n} \beta_{j,i} \kappa_j e^{(\lambda_j + \theta_j)t}, \\
C(t) &= \sum_{j=1}^{n} \sum_{i=1}^{n} \beta_{j,i} \mu_i e^{(\lambda_j - \theta_i)t}, \\
D(t, u) &= \sum_{j=1}^{n} \sum_{i=1}^{n} \beta_{j,i} \bar{\varphi}(u) e^{(\lambda_j - \psi(u))t}.
\end{align*}
\]

**Remark 3.4.** (a) From Proposition 3.4, we can easily obtain an expression for the integrals \( \int_0^t B(s)c(s)ds \) and \( \int_0^t C(s)D(s)ds \) that are involved in Propositions 3.2 and 3.3, as shown below.

\[
\int_0^t B(s)c(s)ds = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \beta_{1,i_2} \kappa_{i_2} \beta_{i_3,i_4} \mu_{i_4}
\times H(t; \lambda_{i_1} + \theta_{i_2} + \psi_{i_4}(u)),
\]

and

\[
\int_0^t C(s)D(s)ds = \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \sum_{i_4=1}^{n} \beta_{i_2,i_3} \kappa_{i_2} \beta_{i_4,i_5} \bar{\varphi}(u)
\times H(t; \lambda_{i_2} - \mu_{i_2} + \psi_{i_3}(u)),
\]

where

\[
H(t; x) := \begin{cases} 
\frac{1}{\gamma} (e^{\gamma x} - 1), & \text{if } x \neq 0; \\
\frac{1}{\gamma}, & \text{if } x = 0.
\end{cases}
\]

(b) By comparing Eq. (3.12) with Eq. (3.24), we notice that expressions for function \( A(t) \) and the expected gain \( EG \) are in the same form. Thus, the expression for \( A(t) \) established in Proposition 3.4 implies

\[
EG(T) = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{i,j} e^{(\lambda_j + \theta_j)t}.
\]

**4. Implementation under some popular exponential Lévy models**

The results we established in Propositions 3.1 and 3.2 are expressed in terms of functions \( A(t), B(t) \), which are respectively defined in (3.12), (3.22) and (3.23). As shown in the previous section, these three functions further depend on quantities \( \mu_j, k_j \) and \( \theta_j \), which are respectively given in (3.17), (3.19) and (3.20). Therefore, to implement our results, we essentially need to calculate these three quantities for a given exponential Lévy process of the asset price. In this section, we will study their calculation for some popular exponential Lévy processes including the Merton’s jump–diffusion, Kou’s jump–diffusion, variance gamma, and normal inverse Gaussian models. As we will see shortly, the calculation is rather amenable for the first two models, but it is quite computational demanding for the last two.

Prevailing financial models with jumps can be basically categorized into two groups. In the first group are the so-called jump–diffusion models, where the “normal” evolution of prices is given by a diffusion process and punctuated by jumps at random intervals. Merton’s model and Kou’s model are the most popular two among this group. The second group consists of the so-called infinite activity models as they admit infinite number of jumps in every interval of time. Many models from the second group can be constructed via Brownian subordination, and both the variance gamma model and the normal inverse Gaussian model fall into this group. We refer the reader to Schoutens (2003), Cont and Tankov (2004) and references therein for an introduction to a variety of interesting Lévy processes as well as their applications in finance.

For simplicity, hereafter we assume that the bond price process \( B_t = B_0 e^{rt} \) with a constant interest rate \( r \) so that the driven process \( R_t = rt, t \geq 0; \) see (2.3). Furthermore, we assume that, conditional on event \( \{X_t = \xi, t \in (u, v]\} \), the stock index admits the following dynamics

\[
S_t = S_u \exp \left( M_t^0 - M_u^0 \right)
\] (4.39)

for \( n \) independent Lévy processes \( \{M_t^j, t \geq 0\}, j = 1, \ldots, n \). We shall investigate how to compute \( \mu_j, k_j \) and \( \theta_j \) for a given Lévy process \( M_t^j \) with known Lévy triplet.

**4.1. Merton’s model**

Merton’s model is the first proposed to address the option pricing problem when the underlying asset admits price jumps; see Merton (1976). Merton’s model (with the regime switching feature under consideration) assumes the following form for the logarithmic price \( M_t^0 \):

\[
M_t^0 = \left( a_t - \frac{\sigma_t^2}{2} \right) t + \sigma_t W(t) + \sum_{i=1}^{N_t(t)} Y_t^i,
\] (4.40)

where \( a_t \in \mathbb{R}, \sigma_t > 0, W(t) \) is a standard Brownian motion, \( N_t(t) \) is a Poisson process with intensity rate \( \lambda_t^\rho \), and \( \{Y_1^i, Y_2^i, \ldots\} \)
is a sequence of independent and identically distributed normal random variables subject to a density function
\[ f_j(x) = \frac{1}{\sqrt{2\pi \delta_j^2}} e^{-\frac{(x - c_j)^2}{2\delta_j^2}} , \]
with mean \( c_j \) and variance \( \delta_j^2 \).

Applying part (b) of Lemma 3.2 yields the following form for the driving process \( Z_t^{(i)} \) of the stock index process:
\[ Z_t^{(i)} = a_t + \sigma_t W(t) + \sum_{j=1}^{N_j(t)} \left( e^{\eta_j} - 1 \right) , \tag{4.41} \]
see (2.2) and (2.3) for definition of \( Z_t^{(i)} \). Consequently, the driving process \( L_t^{(i)} \) defined in (2.8) has a drift \( b_t = a_t - r \), a volatility \( \sigma_t \) and a Lévy measure
\[ \nu_j(dx) = \frac{\lambda^*}{(x + 1)^{\frac{3}{2}}} \exp \left( -\frac{\ln(x + 1) - c_j^2}{2\delta_j^2} \right) dx, \]
\[ x > -1. \]

From the Lévy triplet \((\sigma_t^2, \nu_j, b_j)\) obtained in the above for \( L_t^{(i)} \), we can use (3.17), (3.19) and (3.20), and integrate directly to obtain expressions of \( \mu_j, \kappa_j \) and \( \theta_j \) as given below:
\[ \mu_j = \int_{x \leq -1/m} \nu_j(dx) = \lambda^* h(c_j, \delta_j), \]
\[ \kappa_j = 1 + \frac{m}{\delta_j} \int_{-1/m}^{1/m} x \nu_j(dx) \]
\[ = m\delta_j^2 + 2h(c_j, \delta_j) \]
\[ - (1 - h(c_j, \delta_j)) \]
\[ \theta_j = mb_j + m \int_{x \leq -1/m} x \nu_j(dx) \]
\[ = mb_j + m\lambda^* \left\{ e^{\eta_j} + \frac{\delta_j^2}{2} \left( 1 - h(c_j + \delta_j, \delta_j) \right) \right\} \]
\[ - (1 - h(c_j, \delta_j)) \]
where
\[ h(c, \delta) = \Phi \left( \frac{\ln(1 - 1/m) - c}{\delta} \right), \quad c \in \mathbb{R}, \delta > 0, \]
and \( \Phi(\cdot) \) denotes the standard normal distribution function.

4.2. Kou’s model

Kou’s model, also known as the double exponential jump-diffusion model, admits the same structure as Merton’s model introduced in the last subsection. The only difference lies in the assumption on jump distribution. While the jumps are assumed to have a normal distribution in Merton’s model, they have a double exponential distribution in Kou’s model. Specifically, Kou’s model assumes that the logarithmic price process \( M_t^{(i)} \) of the stock index is as defined in (4.40) with all the other components being the same as in Merton’s model while the jumps \( \{ Y_1^{(i)}, Y_2^{(i)}, \ldots \} \) being a sequence of independent and identically distributed random variables subject to the following double exponential density function
\[ f_j(x) = \frac{p_j}{\eta_j} e^{-p_j \left| x / \eta_j \right|} I_{x > 0} + \frac{q_j}{\eta_j} e^{-q_j \left| x / \eta_j \right|} I_{x < 0} . \tag{4.42} \]
where parameters \( \eta_j^\pm > 0 \) and \( p_j + q_j = 1 \) for \( j = 1, \ldots, n \). The memoryless property of the exponential distribution makes Kou’s model much more tractable relative to the other jump-diffusion models in many situations. As such, Kou’s model has been widely applied in the literature for option pricing and hedging; see, for example, Kou (2002) and Kou and Wang (2004) among many others.

Following the same reasoning stream as we did for Merton’s model in the last subsection, we can easily show that the driving process \( L_t^{(i)} \) defined in (2.8) has a drift \( b_j = a_j - r \), a volatility \( \sigma_j \) and a Lévy measure
\[ \nu_j(dx) = \frac{\delta_j^+}{(x + 1)^{1 - \delta_j^+}} I_{x > 0} \]
\[ + \frac{\delta_j^-}{(x + 1)^{1 + \delta_j^-}} I_{x < 0} dx, \]
where \( \delta_j^+ = 1/\eta_j^+, \delta_j^- = p_j\lambda_j^+ \), and \( \delta_j^- = q_j\lambda_j^- \). From the Lévy triplet \((\sigma_j^2, \nu_j, b_j)\) obtained in the above for \( L_t^{(i)} \), we can use (3.17), (3.19) and (3.20), and integrate directly to obtain expressions of \( \mu_j, \kappa_j \) and \( \theta_j \) as given below:
\[ \mu_j = \int_{x \leq -1/m} \nu_j(dx) = c_j^+ \left( 1 - \frac{1}{m} \right) \]
\[ \kappa_j = 1 + \frac{m}{\delta_j^+} \int_{-1/m}^{1/m} x \nu_j(dx) = -m \frac{1}{\delta_j^-} + 1, \]
\[ \theta_j = mb_j + m \int_{x \leq -1/m} x \nu_j(dx) \]
\[ = mb_j + m\lambda^* \left\{ e^{\eta_j} + \frac{\delta_j^2}{2} \left( 1 - h(c_j + \delta_j, \delta_j) \right) \right\} \]
\[ - (1 - h(c_j, \delta_j)) \]
\[ - 1 - \delta_j^- \left( 1 - \frac{1}{m} \right) \]
\[ - \delta_j^+ . \tag{4.45} \]

4.3. Variance gamma process

The variance gamma (VG) model is one of the most popular infinite activity models with finite variation but relatively low activity of small jumps. It was introduced by Madan and Seneta (1990) as a model for stock returns; for its use in option pricing, we refer the reader to Madan and Milne (1991) and Madan et al. (1998) among many others. It is an important special case of the CGMY distribution proposed by Carr et al. (2002); also see Schoutens (2003) and Cont and Tankov (2004). The VG model assumes the following dynamic for the logarithmic price processes
\[ M_t^{(i)} = m_t + \omega_t \Gamma(t; 1, \delta_j) + \sigma_t W(\Gamma(t; 1, \delta_j)), \quad t \geq 0, \tag{4.46} \]
where \( m_t \in \mathbb{R}, \omega_t \in \mathbb{R}, \sigma_t > 0, \Gamma(t; u, v) \) denotes a gamma process with mean rate \( u \) and variance rate \( v \), and \( W \) is a standard Brownian motion. A gamma process is a Lévy process with independent gamma increments over non-overlapping intervals of time.

\[ M_t^{(i)} \] defined in (4.46) has a Lévy triplet of \((0, v^+_j, m_j)\) with Lévy measure
\[ \nu_j(dx) = \frac{1}{\delta_j^+ |x|} \exp \left( A_j x - B_j |x| \right) dx, \quad x \in \mathbb{R} \setminus \{0\}, \tag{4.47} \]
and characteristic function
\[ \mathbb{E} \left( e^{i \omega M_t^{(i)}} \right) = e^{i \omega M_t^{(i)}} \left( 1 - i \omega \alpha_j \delta_j + \frac{\alpha_j^2 \sigma_j^2}{2} \delta_j \right)^{-t/\delta_j} \tag{4.48} \]
where \( i = \sqrt{-1} \) is the unit imaginary number,
\[ A_j = \frac{\omega_j}{\sigma_j^2} \quad \text{and} \quad B_j = \frac{\sqrt{\omega_j^2 + 2\sigma_j^2/\delta_j}}{\sigma_j^2}; \]
for its derivation, see for example Revuz and Yor (1999), Schoutens (2003) and Cont and Tankov (2004). We note that \( B_j > A_j \), and moreover, (4.48) implies that \( E \left[ e^{\delta_1 t} \right] < \infty \) if and only if \( 1 > \omega_0 \delta_1 + \sigma^2 \delta_1 / 2 \), which is equivalent to the condition \( B_j > A_j + 1 \). We will have to impose this condition for the value of \( \theta_j \); otherwise, \( \theta_j \) in (3.18) is ill-defined.

Using part (b) of Lemma 3.2, we can easily see that the diffusion coefficient for \( L^{(j)} \) is zero and its Lévy measure (which is the same as that of \( Z^{(j)} \)) is as follows

\[
v_j(dx) = \frac{1}{v} \left( 1 + x \right)^{\beta_j - 1} e^{-(B_j + A_j) \ln (1 + x)} \left( \ln (1 + x) \right)^{-v_j} \, dx, \quad x > -1. \tag{4.49}
\]

The drift coefficient for \( L^{(j)} \) should be computed by the following formula

\[
b_j = m_j - r + \int \left( \left( e^x - 1 \right) \right) \, v^*_j(dx) \times \left( e^x - 1 \right) \left( \frac{1}{v} \right) \left( \ln (1 + x) \right)^{-v_j} \, dx.
\tag{4.50}
\]

By resorting to the zero truncation function for calculating the integral in (4.50), we can obtain an expression for \( b_j \) analytically as follows

\[
b_j = m_j - r - \frac{1}{\delta_j} \left( \frac{2A_j}{B_j^2 - A_j} + \frac{e^{-(B_j + A_j)}}{B_j + A_j} \right)
- \frac{e^{-(B_j - A_j)}}{B_j - A_j} - E_i((A_j - B_j) \ln 2) \right) + C_j,
\tag{4.51}
\]

for a constant

\[
C_j = \begin{cases} 
\frac{1}{\delta_j} \ln \left( \frac{B_j^2 - A_j^2}{(B_j - A_j - 1)(B_j + A_j + 1)} \right), & \text{if } B_j \neq A_j + 1, \\
+ \frac{1}{\delta_j} E_i((1 + A_j - B_j) \ln 2), & \text{if } B_j = A_j + 1,
\end{cases}
\]

where

\[
E_i(x) = -\int_{-\infty}^{\infty} e^{-t} \frac{dt}{t} \tag{4.52}
\]

is known as the exponential integral function, and \( \gamma \) is the Euler–Mascheroni constant (also known as the Euler constant) defined by

\[
\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right),
\]

approximately equal to 0.57721. The proof of Eq. (4.51) is relegated to Appendix, and it involves applying the zero truncation function for evaluating the integral in (4.50).

From the Lévy triple of \( L^{(j)} \) we obtained in the above, we apply formulas (3.17), (3.19), and (3.20) to obtain expressions for \( \mu_j, \kappa_j \) and \( \theta_j \) as follows:

\[
\mu_j = \int_{x \leq -1/m} v_j(dx) = -\frac{1}{v} \frac{1}{\delta_j} \left( (A_j + B_j) \ln \left( 1 - \frac{1}{m} \right) \right).
\]

\[
\kappa_j = \frac{1}{\mu_j} \int_{x \leq -1/m} x \, v_j(dx)
= 1 + \frac{1}{\delta_j} \int_{x \leq -1/m} \left( \frac{e^{-(B_j + A_j + 1)}}{B_j + A_j + 1} \right) \ln \left( 1 - \frac{1}{m} \right) \\
- E_i((A_j + B_j) \ln \left( 1 - \frac{1}{m} \right)) \right) + H(A_j + B_j) - H(A_j + B_j + 1) \right),
\]

where

\[
\theta_j = \frac{\mu_j m}{\delta_j} \int_{x \leq -1/m} x \, v_j(dx)
= \frac{m}{\delta_j} \left[ \ln \left( \frac{B_j^2 - (A_j + 1)^2}{B_j^2 - A_j^2} \right) \right]
+ H(A_j + B_j) - H(A_j + B_j + 1) \right).
\]

4.4. Normal inverse Gaussian process

The normal inverse Gaussian (NIG) process was introduced by Barndorff-Nielsen (1997, 1998), and studied further by Barndorff-Nielsen and Shephard (2001). It is one of the most popular infinite activity models with infinite variation in any finite interval of time, and it belongs to the class of generalized hyperbolic Lévy processes; see Schoutens (2003) and Cont and Tankov (2004) for more details. The NIG model assumes the following logarithmic price process

\[
\frac{\mu(t)}{\delta} = \alpha t + \beta \delta \frac{\delta}{2} I + \delta W_t,
\]

where \( W \) is a standard Brownian motion and \( I \) is an inverse Gaussian process with parameters \( \alpha \) and \( \beta \) for \( \alpha > 0, -\beta < \beta \) and \( \delta > 0 \). The Lévy measure for the NIG process is given by

\[
v^*_j(dx) = \frac{e^{\delta x} K_0(\alpha |x|)}{\pi |x|} \, dx, \quad x \in \mathbb{R} \setminus \{0\},
\]

where \( K_0(x) \) denotes the modified Bessel function of the third kind with index \( \lambda \); see Appendix A in Schoutens (2003) for the definition. Using part (b) of Lemma 3.2, we can obtain the following Lévy measure for \( L^{(j)} \) (which is the same as that of \( Z^{(j)} \)):

\[
v_j(dx) = \frac{e^{\delta x} K_0(\alpha |x| \ln (1 + x))}{\pi |x| \ln (1 + x)} \, dx, \quad x > -1. \tag{4.53}
\]

where \( K_0(x) \) denotes the modified Bessel function of the third kind with index 1.

In principle, we can follow the same steps as we illustrated for the VG process in the last subsection to calculate the values of \( \mu_j, \kappa_j \) and \( \theta_j \). However, due to the complexity of the Lévy measure \( v_j \) as demonstrated in (4.53), it seems impossible to obtain analytical expressions for these three quantities in the NIG case. We may have to resort to some numerical approach for their values, and we leave this for future exploration.

5. Numerical example

In this section, we shall illustrate how to use Propositions 3.1 and 3.2 to calculate these quantities including shortfall probability, expected shortfall and expected gain under a specific risk asset price model. We consider Kou’s model, which is also called the double exponential jump–diffusion model, as an example; see Section 4.2 for more details. As we can see shortly, the existence of the regime switching feature has a significant effect on the performance of the CPPI portfolio. A different market state at the inception of the investment will lead to a quite different risk–reward profile of a CPPI portfolio.
Fig. 1. Shortfall probability over $T = 2$ years as a function of the multiplier.

Under Kou’s model, the logarithmic price process $M_t^{(j)}$ and the driving processes $Z_t^{(j)}$ respectively admit a form as in (4.40) and (4.41), with double exponentially distributed jumps $\{Y_t^{(j)}\}$ as explained in Section 4.2. The density function of the jumps is given in (4.42). Analytic expressions for $\mu_j$, $\kappa_j$ and $\theta_j$ are respectively given in (3.17), (3.19) and (3.20).

We let $n = 2$ meaning that the market is switching between two states—bull and bear, and the intensity matrix of the regime switching Markov process is assumed to be

$$Q = \begin{pmatrix}
-0.25 & 0.25 \\
0.25 & -0.25
\end{pmatrix}, \quad (5.54)$$

which implies that it is equally likely for the market to change from a bull to a bear and from a bear to a bull. Under each market regime, parameter values of the model are set as shown in Table 1, where the two regimes are distinguished by a positive expected return in a bull market and a negative one in a bear market. Moreover, the values of parameters $p_j$ are set intentionally to reflect the fact that the stock price is more likely to jump upwards than downwards in a bull market and to jump downwards than upwards in a bear market. Furthermore, the values of $\eta^-$ imply that the downward jump size is much larger than the upward jump size in a bear market. All of these assumptions are made in accordance to our intuitive observations on the stock market.

To demonstrate the effect from the regime switching feature of the market, we calculate the shortfall probability, (unconditional) expected shortfall and (unconditional) expected gain of the CPPI portfolio in two distinct scenarios: $X_0 = (1, 0)$ and $X_0 = (0, 1)$, which respectively mean that the investment starts with a bull and a bear market state, respectively called “bull-start portfolio” and “bear-start portfolio”. We set $T = 2$, meaning an investment of two years. The results are reported in Figs. 1–3 and Table 2. In addition to the increasing trend of all these quantities as a function of the multiplier $m$, we have the following important observations.

(a) First, the shortfall probability is substantially larger for a bull-start portfolio than a bear-start portfolio; see Fig. 1. In particular, when the multiplier $m$ is as large as 7, there is a probability of 3.3919% to realize a shortfall within the investment of one year if the CPPI portfolio starts in a bear market, while it is only about 1.7256% if it starts in a bull market.

(b) Second, the unconditional expected shortfall is larger for a bear-start portfolio than a bull-start portfolio for all the chosen multiplier values ($m = 1, 2, \ldots, 7$); see the left graph in Fig. 2. Nevertheless, the conditional expected shortfall does not show such a strict ordering phenomenon between the bull-start portfolio and the bear-start portfolio. The right graph in Fig. 2 indicates that the conditional expected shortfall is slightly smaller for the bull-start portfolio than the bear-start portfolio when the multiplier $m$ is as small as 3. For a multiplier of no less than 4, however, the conditional expected shortfall is always larger for the bull-start portfolio than the bear-start portfolio, and the gap increases rapidly as the multiplier $m$ increases. These observations indicate that the risk on the guarantor substantially differs from a bull-start

Table 1
<table>
<thead>
<tr>
<th>Parameter values.</th>
<th>$a_j$</th>
<th>$r$</th>
<th>$b_j$</th>
<th>$\sigma_j$</th>
<th>$\lambda_j^+$</th>
<th>$p_j$</th>
<th>$\eta_j^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bull ($j = 1$)</td>
<td>0.14</td>
<td>0.04</td>
<td>0.31</td>
<td>0.16</td>
<td>0.55</td>
<td>0.013</td>
<td>0.016</td>
</tr>
<tr>
<td>Bear ($j = 2$)</td>
<td>-0.01</td>
<td>0.04</td>
<td>-0.05</td>
<td>0.20</td>
<td>0.50</td>
<td>0.100</td>
<td>0.013</td>
</tr>
</tbody>
</table>

Table 2
<table>
<thead>
<tr>
<th>Comparison between an initial market state of bull and bear.</th>
<th>$m$</th>
<th>SF (%)</th>
<th>UES</th>
<th>ES</th>
<th>UEG</th>
<th>EG</th>
</tr>
</thead>
<tbody>
<tr>
<td>SF (%)</td>
<td>0.0020</td>
<td>0.0523</td>
<td>0.4144</td>
<td>1.7256</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UES</td>
<td>0.0008</td>
<td>0.0314</td>
<td>0.3663</td>
<td>2.1532</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ES</td>
<td>38.3959</td>
<td>60.0430</td>
<td>88.4145</td>
<td>124.7812</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UEG</td>
<td>0.0328</td>
<td>1.0525</td>
<td>10.3496</td>
<td>53.6167</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EG</td>
<td>1625.5057</td>
<td>2013.1046</td>
<td>2497.7835</td>
<td>3107.0973</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SF (%)</td>
<td>0.0043</td>
<td>0.1085</td>
<td>0.8377</td>
<td>3.3919</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UES</td>
<td>0.0014</td>
<td>0.0481</td>
<td>0.4794</td>
<td>2.4747</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ES</td>
<td>33.4657</td>
<td>44.3524</td>
<td>57.2314</td>
<td>72.9600</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UEG</td>
<td>0.0325</td>
<td>1.0125</td>
<td>9.6769</td>
<td>48.7133</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EG</td>
<td>758.5772</td>
<td>932.8299</td>
<td>1155.1486</td>
<td>1436.1573</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 2. Expected shortfall over $T = 2$ years as a function of the multiplier, for $C_0^* = $1000; on the left is the unconditional expected shortfall while, on the right, is the expected shortfall conditional on a shortfall having occurred.
portfolio to a bear-start portfolio. All in all, there is more risk on the CPPI fund guarantor if the investors choose to enter the fund in a bull market than in a bear market, which in turn implies that more premium should be charged on the investors if they choose to enter the fund in a bear market.

(c) Third, the risk–reward profile for the investors can also be quite different between starting the investment in a bull market and a bear market; see Fig. 3. Consider the case with \( m = 6 \) for an example; see Table 2. If an investor entered the CPPI fund in a bull market, there is a probability of 0.4144% for him/her to obtain a payoff at the guarantee level, and a very large probability of 99.5856% to receive an expected gain of $2497.78. On the contrary, if the investor started in a bear market, the probability for him/her to receive a payoff at the guarantee level is 0.8377%, and given no shortfall having occurred his/her expected gain will be $1155.15. Although the unconditional expected gain is closed, the risk-reward profile is quite different between these two types of CPPI portfolios.

6. Conclusion

The regime switching framework for modeling econometric series offers a transparent and intuitive way to capture market behavior through different economic conditions. It has been widely used in econometrics since the pioneering work of Hamilton (1989). The present paper developed a framework for studying the performance of the CPPI portfolio in the presence of jumps in the asset price and the regime switching feature of the financial market. Analytically tractable expressions for the shortfall probability, expected shortfall, expected gain and the characteristic function of the “gap risk” are derived under a general Markov regime switching exponential Lévy model. These results can not only help the investors to develop a comprehensive understanding on their risk-and-reward profiles, but also offer an effective framework for CPPI fund guarantors to conduct stress tests by adjusting input parameters. Our results show that the behavior of the CPPI portfolio can differ significantly with a different market state at the inception of the investment. Indeed, from a perspective of the guarantor, it is much riskier if the portfolio starts in a bull market than in a bear market.

Appendix

Proof of Lemma 3.4. Clearly, the first equation in (3.36) with the initial condition \( r_1(t) = 1 \) implies \( r_1(t) = e^{r_1 t} \). Substituting \( r_1(t) = e^{r_1 t} \) into its second equation \( \frac{d}{dt} r_2(t) = r_1(t) + \lambda_2 r_2(t) \), and solving it for \( r_2(t) \) with the initial condition \( r_2(0) = 1 \), we obtain

\[
    r_2(t) = e^{r_1 t} \frac{1}{\lambda_1 - \lambda_2} + e^{r_1 t} \frac{1}{\lambda_2 - \lambda_1}. 
\]

This implies that the expression (3.37) holds for \( n = 2 \). We will complete the proof by the induction principle.

Suppose (3.37) is true for some positive integer \( n \). Then, from (3.36), \( \rho_{n+1}(t) \) satisfies the following differential equation:

\[
    \frac{d}{dt} \rho_{n+1}(t) = \sum_{j=1}^{n+1} \left( e^{r_1 j t} \prod_{k=1,k \neq j}^{n} \frac{1}{\lambda_j - \lambda_k} \right) + \lambda_{n+1} \rho_{n+1}(t). \quad (A.1)
\]

Thus, it remains to show that

\[
    \rho_{n+1}(t) = \sum_{j=1}^{n+1} \left( e^{r_1 j t} \prod_{k=1,k \neq j}^{n+1} \frac{1}{\lambda_j - \lambda_k} \right) \quad (A.2)
\]

satisfies Eq. (A.1). In fact, on the one hand, (A.2) implies

\[
    \frac{d}{dt} \rho_{n+1}(t) = \sum_{j=1}^{n+1} \left( \lambda_{n+1} e^{r_1 j t} \prod_{k=1,k \neq j}^{n+1} \frac{1}{\lambda_j - \lambda_k} \right). \quad (A.3)
\]

On the other hand, the right hand side of (A.1)

\[
    \sum_{j=1}^{n} \left( e^{r_1 j t} \prod_{k=1,k \neq j}^{n} \frac{1}{\lambda_j - \lambda_k} \right) + \lambda_{n+1} \rho_{n+1}(t)
\]

\[
    = \sum_{j=1}^{n} \left( e^{r_1 j t} \prod_{k=1,k \neq j}^{n} \frac{1}{\lambda_j - \lambda_k} \right)
\]

\[
    + \lambda_{n+1} \sum_{j=1}^{n+1} \left( e^{r_1 j t} \prod_{k=1,k \neq j}^{n+1} \frac{1}{\lambda_j - \lambda_k} \right)
\]

\[
    = \sum_{j=1}^{n} \left[ e^{r_1 j t} \left( \prod_{k=1,k \neq j}^{n} \frac{1}{\lambda_j - \lambda_k} + \lambda_{n+1} \prod_{k=1,k \neq j}^{n+1} \frac{1}{\lambda_j - \lambda_k} \right) \right]
\]

\[
    + \lambda_{n+1} \left( e^{r_1 j t} \prod_{k=1,k \neq j}^{n+1} \frac{1}{\lambda_j - \lambda_k} \right)
\]

\[
    = \sum_{j=1}^{n} \left[ \left( e^{r_1 j t} \prod_{k=1,k \neq j}^{n} \frac{1}{\lambda_j - \lambda_k} + \lambda_{n+1} \prod_{k=1,k \neq j}^{n+1} \frac{1}{\lambda_j - \lambda_k} \right) \right]
\]

\[
    + \lambda_{n+1} \left( e^{r_1 j t} \prod_{k=1,k \neq j}^{n+1} \frac{1}{\lambda_j - \lambda_k} \right)
\]

Acknowledgments

The author acknowledges the funding support from the University of Waterloo Start-up Grant, and the Natural Sciences and Engineering Research Council of Canada (NSERC).
\[
\lambda_{n+1} e^{\frac{1}{\lambda_{n+1}}} + \lambda_{n+1} e^{\frac{1}{\lambda_n}} + \sum_{j=1}^{n+1} \left( \frac{1}{\lambda_j - \lambda_k} \right) e^{\frac{1}{\lambda_j - \lambda_k}} + \sum_{j=1}^{n+1} \left( \frac{1}{\lambda_j - \lambda_k} \right) e^{\frac{1}{\lambda_j - \lambda_k}}
\]

which implies that the desired result holds for \( n + 1 \) as well, and thus, by the induction principle, the proof is complete. \( \square \)

**Proof of Proposition 3.4.** We only show the expression for function \( A \), as the others can be derived in a similar way. We first note that \( e^{Q - \text{diag}(j\mu)} = e^{Q_1 e^{-\text{diag}(j\mu)}}, \) and for a diagonal matrix,

\[
\exp(-\text{diag}(\mu)) = \begin{pmatrix} e^{-t\mu_1}, & 0, & \cdots & 0 \\
0, & e^{-2t\mu_2}, & \cdots & 0 \\
0, & 0, & \cdots & e^{-nt\mu_n} \\
0, & 0, & \cdots & 0 \\
\end{pmatrix}
\]

Thus, combining Eq. (3.12) and Lemma 3.5 yields

\[
A(t) = \{1, (\mathbb{E}(X_0))', e^{Q - \text{diag}(j\mu)} \} = (\mathbb{E}(X_0))' e^{Q_1 e^{-\text{diag}(j\mu)} 1}
\]

\[=
(\mathbb{E}(X_0))' e^{Q_1 e^{-\text{diag}(j\mu)} 1}
\]

as desired. \( \square \)

**Proof of Eq. (4.51).** To save lines, we only show the result for the case with \( B > A + 1 \), and the results follow in a completely parallel way for the other cases. From (4.47) and (4.50), we have

\[
b_j = m_j - r + \lim_{\epsilon \to 0^+} \{ I_{11}(\epsilon) + I_{12}(\epsilon) \} + I_2,
\]

where

\[
I_{11}(\epsilon) = - \int_{-\infty}^{-\epsilon} (e^{x} - 1) \frac{1}{\delta x} e^{(A_j + B_j)x} dx,
\]

\[
I_{12}(\epsilon) = \int_{\epsilon}^{in^2} (e^{x} - 1) \frac{1}{\delta x} e^{-(B_j - A_j)x} dx,
\]

and

\[
I_2 = \int_{-\frac{1}{\delta x}}^{1} \frac{x}{x} e^{(B_j - A_j)|x|} dx.
\]

Using the fact that \( B_j > A_j \), we can directly calculate the integral for \( I_2 \) to obtain

\[
I_2 = \frac{1}{\delta_j} \left\{ 2A_j + \left( e^{-(B_j + A_j)} - e^{-(B_j - A_j)} \right) \right\}.
\]

To proceed, we recall the definition of exponential integral function \( Ei \) in (4.52) and the following expression for \( Ei \):

\[Ei(x) = \gamma + \ln |x| + \sum_{k=1}^{\infty} \frac{x^k}{(k)!}, \quad x \neq 0,
\]
where $\gamma$ is the Euler constant. We also note that $\lim_{t \to 0} \sum_{k=1}^{\infty} \frac{e^{x}}{\Gamma(k+1)} = 0$. Consequently, by applying these facts and the change-of-variable formula for the integral, we obtain

$$
\lim_{\epsilon \to 0^+} I_{11}(\epsilon) = \lim_{\epsilon \to 0^+} \left( \int_{(B_j + A_j + 1)\epsilon}^{\infty} e^{-u} \frac{e^{-u}}{u} du - \int_{(B_j + A_j)\epsilon}^{\infty} e^{-u} \frac{e^{-u}}{u} du \right)
$$

$$
= \lim_{\epsilon \to 0^+} \left\{ -\text{Ei}(-((B_j + A_j + 1)\epsilon)) - \text{Ei}(-((B_j + A_j)\epsilon)) \right\}
$$

$$
= \ln \left( \frac{B_j + A_j}{B_j + A_j + 1} \right), \quad (A.7)
$$

and

$$
\lim_{\epsilon \to 0^+} I_{12}(\epsilon) = \lim_{\epsilon \to 0^+} \left\{ \int_{(B_j - A_j - 1)\epsilon}^{\infty} e^{-u} \frac{e^{-u}}{u} du - \int_{(B_j - A_j)\epsilon}^{\infty} e^{-u} \frac{e^{-u}}{u} du \right\}
$$

$$
= \lim_{\epsilon \to 0^+} \left\{ -\text{Ei}(-((B_j - A_j - 1)\epsilon)) + \text{Ei}(-((B_j - A_j)\epsilon)) + \text{Ei}(-(B_j - A_j - 1)\epsilon) - \text{Ei}(-(B_j - A_j)\epsilon) \right\}
$$

$$
= \ln \left( \frac{B_j - A_j}{B_j - A_j - 1} \right) + \text{Ei}(-(B_j - A_j - 1)\epsilon) - \text{Ei}(-(B_j - A_j)\epsilon) \right\}, \quad (A.8)
$$

Combining (A.5)–(A.8) yields the desired result. \(
\square
\)

References


